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PROPERTIES OF DISCONTINUOUS BIFURCATION SOLUTIONS IN ELASTO-PLASTICITY

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Abstract—Explicit expressions for the spectral properties of the bifurcation problem involving discontinuities for general elastic–plastic materials are presented. It then follows that the classical value of the critical hardening modulus derived by Rice (1976, *Proc. 14th IUTAM Congr., Delft, The Netherlands*, pp. 207–220. North Holland, Amsterdam.) is, in fact, the only possible one. Furthermore, from the spectral analysis it follows in a very straightforward fashion that bifurcation displaying elastic unloading on one side of the singular surface can never precede bifurcation with plastic loading on both sides of this surface. Explicit analytical results for the critical bifurcation directions and the corresponding hardening modulus are derived for non-associated volumetric flow rules while the deviatoric portion is associated. The considered yield and potential functions may depend on all three stress invariants and may involve mixed isotropic and kinematic hardening. The result obtained by Rudnicki and Rice (1975, *J. Mech. Phys. Solids* **23**, 371–394) for a Drucker–Prager material appear as a special case. Other criteria that are investigated are those of Coulomb and Rankine.

1. INTRODUCTION

A vast amount of work has been devoted to explaining and characterizing the abrupt changes in the deformation field that occur in elastic–plastic bodies across narrow zones (shear bands). The existence of Lüders bands in thin steel plates subjected to uniaxial stress states, e.g. Nadai (1950), represents a classical example of such behavior. In the pioneering works of Rice (1976) and Rudnicki and Rice (1975), the phenomenon was attributed directly to the properties of the constitutive model, although geometric effects were also included. Several papers, e.g. Hill and Hutchinson (1975), Stören and Rice (1975), have particularly addressed the combined constitutive and geometric effects for the necking problem.

The concept of a "characteristic surface", across which bifurcation is permitted not only of the rate of deformation gradient but also of the velocity itself, is used in this paper. This approach, which was considered already by Hill (1950) and Thomas (1961), thus includes the possibility of "infinitely localized" deformation, although the difference to the "shear band approach" used in Rice (1976) and Rudnicki and Rice (1975) is subtle since the resulting kinematic restrictions are precisely the same. At this point it may be remarked that the "classical" arguments for localization refer only to a discontinuity in the rate of the deformation gradient within the shear band, which implies that the shear band is located between two (or more) characteristic surfaces.

Arguments for utilizing the concept of a discontinuous velocity field have been put forward elsewhere. For example, in concentration with thermodynamical considerations it was shown in Ottosen (1986) and Barré and Maier (1988) that, for the simple example of a concrete bar loaded in tension into its post-peak region, the width of the localization zone will decrease to zero thereby violating the classical definition of strain. However, especially for granular materials there seems to be ample experimental evidence that the localization zone does, indeed, have finite width. In order to determine this band width in bodies of

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sand, Mühlhaus and Vardoulakis (1987) used micromechanics in terms of nonlocal Cosserat theory.

In this paper we shall first establish analytical expressions for the spectral properties of the characteristic tangent stiffness tensor which in a straightforward fashion enables us to make some general conclusions regarding discontinuous bifurcations. Moreover, this spectral analysis is also useful in establishing explicit relations for the polarized wave speeds of the related acceleration wave problem, cf. Ottosen and Runesson (1990). The main focus of this paper is, however, on the analytical determination of the critical bifurcation directions and the corresponding critical value of the hardening modulus. These solutions generalize the results obtained by Rudnicki and Rice (1975) to a broader class of plasticity models.

A major motivation to determine analytically the critical bifurcation directions is the interest in localization phenomena that has been displayed in recent years in the context of finite element analysis. To capture localized deformation patterns along the characteristics, the element mesh may be designed (and aligned) accordingly or enriched with special shape functions as described by, e.g. Ortiz *et al.* (1987). It is clear that the deformation field is significantly constrained for an arbitrarily chosen mesh, especially when the velocity field is continuous along the element boundaries. Another possibility is, therefore, to use discontinuous shape functions to enhance the non-smooth character of the plastic solution as suggested by Johnson and Scott (1981).

In the general case it is, of course, possible to employ a numerical search algorithm in order to find a maximum (critical) value of the hardening modulus. This strategy was adopted by Ortiz *et al.* (1987). However, it is desirable to avoid such a numerical search for two reasons: Firstly, the search effort is time consuming, since a large number of material points must be considered; typically the nodes or the Gaussian integration points. Secondly, the local maximum that is detected via the search algorithm may not be the true (global) maximum as multiple extrema will in general be present, as shown later. It is therefore of considerable interest to obtain analytical expressions for the critical bifurcation directions and the corresponding maximum value of the hardening modulus for a larger class of elastic–plastic models than that treated by Rudnicki and Rice (1975).

Subsequently, if not otherwise is explicitly stated, the notion of bifurcation solution refers to an incremental solution involving a discontinuity.

2. CONSTITUTIVE FORMULATION OF PLASTICITY

For simplicity it will be assumed that deformations are small, i.e. configuration changes are neglected. The latter assumption is introduced for simplicity, although it is straightforward to generalize the formulation by introducing an objective rate of Cauchy stress (such as the corotational rate) rather than the ordinary time rate. With σ_{ij} and ε_{ij} being the Cartesian components of the stress and strain tensors respectively, and a superimposed dot denoting time rate, we assume that the constitutive behavior for a nonassociated flow rule is described by the incrementally linear relationship

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\varepsilon}_{kl} \tag{1}$$

where the (in general non-symmetric) bilinear tangent stiffness tensor D_{ijkl} is given by

$$D_{ijkl} = \begin{cases} D^{e}_{ijkl} & \text{(E)} \\ D^{e}_{ijkl} - \dfrac{1}{A} D^{e}_{ijst} g_{st} f_{mn} D^{e}_{mnkl} & \text{(P)} \end{cases}, \tag{2}$$

for elastic (E) and plastic (P) loading respectively. D^{e}_{ijkl} are the components of the elastic stiffness modulus tensor, which is assumed to be constant and symmetric, i.e. $D^{e}_{ijkl} = D^{e}_{klij}$, while it may represent isotropic as well as anisotropic behavior. Moreover

$$f_{ij} = \frac{\partial F}{\partial \sigma_{ij}}, \quad g_{ij} = \frac{\partial G}{\partial \sigma_{ij}} \tag{3}$$

where F is the yield function and G is the plastic potential, whose arguments are the stress state and a set of hardening variables. The positive parameter A is defined as

$$A = H + f_{ij} D^e_{ijkl} g_{kl} > 0 \tag{4}$$

where H is the generalized plastic modulus that is positive, zero or negative for hardening, perfect or softening plasticity respectively.

Finally, plastic loading (P) will take place whenever the condition

$$F = 0, \quad \text{and} \quad f_{ij} D^e_{ijkl} \dot{\varepsilon}_{kl} > 0 \tag{5}$$

is satisfied. Otherwise, elastic loading (E) occurs.

3. BIFURCATIONS-CHARACTERISTICS

Assume that the current state of static equilibrium is characterized by continuous displacements u_i, stresses σ_{ij} and strains ε_{ij}. With increased loading we shall consider the possibility that discontinuous bifurcations of the displacement rate $\dot{u}_i$ and the rate of the displacement gradient $\dot{u}_{i,j}$ can occur across a fixed singular surface S within the body. Furthermore, it is assumed that the difference between the value of $\dot{u}_i$ for the bifurcated and primary fields is preserved along S, i.e. $[\dot{u}_i]$ = constant along S, where the bracket denotes discontinuity (difference). Consequently, the strain rate $\dot{\varepsilon}_{ij}$ as well as the stress rate $\dot{\sigma}_{ij}$ become discontinuous across S. We emphasize that no assumption is introduced regarding the homogeneity and variation of $[\dot{u}_{i,j}]$ in a neighbourhood of S, i.e. we have not assumed the existence of a "shear band" of finite thickness, within which $[\dot{u}_{i,j}]$ is constant.

Let the orientation of the characteristic surface S be defined by the unit normal vector n_i and denote the position vector along S by x_i. The assumption that $[\dot{u}_i]$ = constant along S implies that

$$d[\dot{u}_i] = [\dot{u}_{i,j}]\,\mathrm{d}x_j = 0 \tag{6}$$

where $[\dot{u}_{i,j}] = \partial[\dot{u}_i]/\partial x_j$ and $\mathrm{d}x_j$ is an arbitrary differential vector tangential to S. As is well known, the general solution of eqn (6) is

$$[\dot{u}_{i,j}] = c_i n_j, \quad [\dot{\varepsilon}_{ij}] = \tfrac{1}{2}(c_i n_j + c_j n_i) \tag{7}$$

where c_i is an arbitrary vector.

It follows from equilibrium considerations that the traction rate across the singular surface S must be unique

$$[\dot{\sigma}_{ij}] n_j = 0. \tag{8}$$

Suppose now that the material at both sides of the surface S responds plastically (plastic/plastic bifurcation). As the stress and strain states were assumed to be continuous prior to bifurcation, the tangential stiffness tensor D_{ijkl} given by eqn (2) takes the same value on either side of the surface S. Therefore, combining eqns (1), (7) and (8) yields

$$n_j D_{ijkl} n_k c_l = 0 \tag{9}$$

where the symmetry property $D_{ijkl} = D_{ijlk}$ was used.

Consider now the alternative scenario that the material on one side of the surface S responds plastically, while it responds elastically on the other side (elastic/plastic bifurcation). From eqn (8) it then follows that

$$n_j(D_{ijkl}\dot{\varepsilon}''_{kl} - D^e_{ijkl}\dot{\varepsilon}'_{kl}) = 0 \tag{10}$$

where $\dot{\varepsilon}''_{kl}$ corresponds to a plastic response while $\dot{\varepsilon}'_{kl}$ corresponds to an elastic response. Elimination of D^e_{ijkl} by means of eqns (2) and use of eqn (7) results in the relation

$$n_j D_{ijkl} n_k c_l = \frac{1}{A} n_j D^e_{ijst} g_{st} f_{mn} D^e_{mnkl} \dot{\varepsilon}'_{kl} \tag{11}$$

instead of eqn (9). Equations (9) and (11) are the classical bifurcation conditions reviewed by Rice (1976).

Trivial solutions of eqns (9) and (11), i.e. $c_i = 0$, imply that the strain rates $\dot{\varepsilon}_{ij}$ as well as the stress rates $\dot{\sigma}_{ij}$ become unique. As expected, eqn (11) shows that in such a situation the boundary between the elastic and plastic regions is characterized by neutral loading defined as $f_{mn} D^e_{mnkl} \dot{\varepsilon}'_{kl} = 0$.

Consider eqn (9) and define the characteristic tangent stiffness modulus tensor Q_{il} as

$$Q_{il} = n_j D_{ijkl} n_k. \tag{12}$$

Non-trivial solutions of eqn (9) are, of course, possible only when Q_{il} is singular. In analogy with the terminology used to classify scalar second-order partial differential equations (which are defined by a generating matrix representing the entries of Q_{il}), the situation that Q_{il} becomes singular corresponds to loss of ellipticity, e.g. Knops and Payne (1971). Clearly, only bifurcation solutions with the discontinuity defined by eqn (7) are associated with the loss of ellipticity while continuous bifurcations may appear in the elliptic regime as discussed later.

4. SPECTRAL PROPERTIES OF THE CHARACTERISTIC TENSOR

Since the existence of discontinuous bifurcations requires that Q_{il} becomes singular, it is natural to consider the (right) eigenvalue problem

$$Q_{il} y_l^{(i)} = \lambda^{(i)} Q^e_{il} y_l^{(i)}, \quad i = 1, 2, 3 \tag{13}$$

where Q^e_{il} is the characteristic stiffness tensor associated with elastic behavior, i.e.

$$Q^e_{il} = n_j D^e_{ijkl} n_k. \tag{14}$$

It will be shown below that the choice of the "elastic matrix" in eqn (13) makes it simple to find explicitly the eigenvalues $\lambda^{(i)}$ and eigenvectors $y_l^{(i)}$. As D^e_{ijkl} is symmetrical and positive definite, then so is Q^e_{il}, which follows from

$$c_i Q^e_{il} c_l = c_i n_j D^e_{ijkl} n_k c_l = [\dot{\varepsilon}_{ij}] D^e_{ijkl} [\dot{\varepsilon}_{kl}] > 0 \tag{15}$$

where c_i represents a non-zero but arbitrary vector. Therefore, Q^e_{il} possesses the symmetrical and positive definite inverse P^e_{il}. Now, multiply eqn (13) by P^e_{ji} and use eqn (2) to obtain the standard eigenvalue problem

$$B_{jl} y_l^{(i)} = \lambda^{(i)} y_j^{(i)} \tag{16}$$

where B_{jl} is defined by

$$B_{jl} = P^e_{ji} Q_{il} = \delta_{jl} - \frac{1}{A} P^e_{ji} b_i a_l \tag{17}$$

and the vectors a_l and b_i are defined by

$$a_l = f_{mn} D^e_{mnkl} n_k, \quad b_i = n_j D^e_{ijst} g_{st}. \tag{18}$$

It appears that $\lambda = 1$ is an eigenvalue with a multiplicity of two. To show this, we can substitute $\lambda = 1$ into eqn (16) and use eqn (17) to obtain

$$P^e_{ji} b_i a_l y_l = 0. \tag{19}$$

The coefficient matrix in front of y_l can always be written as $q_j a_l$, where $q_j = P^e_{ji} b_i$, i.e. two of the rows are always proportional to the remaining one which implies that $\lambda = 1$ is an eigenvalue with multiplicity two, i.e.

$$\lambda^{(1)} = \lambda^{(2)} = 1, \text{ (elastic modes)}. \tag{20}$$

To find the remaining eigenvalue $\lambda^{(3)}$, we observe from eqns (16) and (20) that

$$B_{jj} = \lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)} = 2 + \lambda^{(3)} \tag{21}$$

which together with eqn (17) gives

$$\lambda^{(3)} = 1 - \frac{1}{A} a_i P^e_{ij} b_j. \tag{22}$$

It is quite straightforward to find the corresponding eigenvectors $y_l^{(i)}$. Since P^e_{ji} is nonsingular, implying that $P^e_{ji} b_i \neq 0$, it follows from eqn (19) that the elastic modes, which correspond to neutral loading, are defined by

$$a_l y_l^{(1)} = a_l y_l^{(2)} = 0. \tag{23}$$

Furthermore, it follows from eqns (16), (17) and (22) that

$$y_j^{(3)} = \gamma P^e_{ji} b_i \tag{24}$$

where γ is an arbitrary constant.

From the spectral properties obtained above it is simple to determine nontrivial solutions to the bifurcation problem (9). These results were obtained previously by Rice (1976) by inspection. The spectral analysis outlined above enables us to prove that the solution obtained in Rice (1976) is, in fact, the only possible one. It appears that there exists only one possibility for a nontrivial solution of eqn (9), namely that $\lambda^{(3)} = 0$. From eqns (4), (18) and (22) we obtain the corresponding value of the hardening modulus

$$H = -f_{ij} D^e_{ijkl} g_{kl} + n_j D^e_{ijst} g_{st} P^e_{il} f_{mn} D^e_{mnkl} n_k \tag{25}$$

in accordance with Rice (1976). The corresponding solution for c_l is given directly by eqn (24) as

$$c_l = \gamma P^e_{li} n_j D^e_{ijst} g_{st} \tag{26}$$

again in accordance with Rice (1976). The task is thus to find the critical value H^{db} corresponding to discontinuous bifurcation, which is defined as the maximum value of H with respect to a variation of the shear band direction n_i for a given state, i.e.

$$H^{db} = \max H(n_i). \tag{27}$$

For isotropic elasticity it was shown in Rice (1976) that $H^{db} \leqslant 0$ always holds for associated plasticity, whereas $H^{db} > 0$ is possible for nonassociated plasticity. We shall see later that this result holds also for anisotropic elasticity. Purely elastic behavior, i.e. $H = \infty$, implies that $A = \infty$ from eqn (4), which in turn infers $\lambda^{(3)} = 1$ from eqn (22). Therefore, in the hardening region where $H > H^{db}$, i.e. $\lambda^{(3)} > 0$, discontinuous bifurcations can never occur.

Let us conclude this section by a further consideration of the elastic/plastic bifurcation condition as expressed by eqn (11). It was shown by Rice and Rudnicki (1980) that this condition is less critical than the plastic/plastic condition (9), i.e. bifurcation involving elastic unloading on one side of the characteristic surface S can never precede bifurcation with plastic loading on both sides of S. However, the spectral analysis provides a straightforward proof, which will be given here for completeness.

From eqns (12) and (18) it follows that eqn (11) can be rewritten as

$$Q_{il}c_l = \frac{\alpha}{A}b_i, \quad \alpha = f_{mn}D^e_{mnkl}\dot{\varepsilon}'_{kl} \leqslant 0 \tag{28}$$

where $\alpha \leqslant 0$ since $\dot{\varepsilon}'_{kl}$ corresponds to elastic unloading. Noting from eqns (7) and (10) that $\dot{\varepsilon}'_{kl} = \dot{\varepsilon}''_{kl} - \frac{1}{2}(c_k n_l + c_l n_k)$, where $\dot{\varepsilon}''_{kl}$ corresponds to plastic loading, we obtain via eqns (12), (14) and (18)

$$Q^e_{il}c_l = \frac{\beta}{A}b_i, \quad \beta = f_{mn}D^e_{mnkl}\dot{\varepsilon}''_{kl} > 0. \tag{29}$$

From eqn (29) we may solve for c_j that represents the non-trivial solution for elastic/plastic bifurcation

$$c_j = \frac{\beta}{A}P^e_{ji}b_i. \tag{30}$$

It is noted that this is the same vector as shown in eqn (24) defining plastic/plastic bifurcation; the only difference being the scaling factor. Now, elimination of b_i between eqns (28) and (29) yields

$$Q_{il}c_l = \frac{\alpha}{\beta}Q^e_{il}c_l \tag{31}$$

which represents an eigenvalue problem completely identical to that given by eqn (13). The "elastic" solutions of eqn (13) defined by $\lambda^{(1)} = \lambda^{(2)} = 1$ can never become relevant for eqn (31), since $\alpha/\beta \leqslant 0$ according to eqns (28) and (29). Therefore, the only possibility for a non-trivial solution of eqn (31) is that

$$\lambda^{(3)} = \frac{\alpha}{\beta} \leqslant 0. \tag{32}$$

As $\lambda^{(3)} \leqslant 0$ this result implies that elastic/plastic bifurcation will never occur before plastic/plastic bifurcation. Only when $\alpha = 0$, which corresponds to neutral loading or $\dot{\varepsilon}'_{ij} = 0$ (no deformation), these two types of conditions coincide as expected from continuity of the incremental response. It is interesting to note that the magnitude of the strain rate discontinuity is linked to the magnitude of $\dot{\varepsilon}''_{ij}$, because c_l is given by eqn (30) and β is defined in terms of $\dot{\varepsilon}''_{ij}$ from eqn (29). In the event of plastic/plastic bifurcation the magnitude of c_l is arbitrary as defined by eqn (26).

At this point it is also worth noting that, after loss of singularity of Q_{il} corresponding to $\lambda^{(3)} < 0$, we have always the possibility for elastic/plastic bifurcations according to eqns (31) and (32), whereas there is no longer any possibility for plastic/plastic bifurcation.

5. GENERAL VERSUS DISCONTINUITIES BIFURCATION

In this section we shall relate the condition for uniqueness, discontinuous bifurcations and limit points. We first note that a sufficient condition for uniqueness is

$$[\dot{\varepsilon}_{ij}]D_{ijkl}[\dot{\varepsilon}_{kl}] > 0 \tag{33}$$

for arbitrary $[\dot{\varepsilon}_{ij}] \neq 0$, i.e. the result holds irrespective of $[\dot{\varepsilon}_{ij}]$ being continuous or discontinuous. We also note that this criterion follows from the assumption of plastic/plastic bifurcation. Let H^b denote the value of the hardening modulus for which eqn (33) is first violated. It is a trivial fact that discontinuous bifurcations (which require restricted kinematics) can never occur before general bifurcation, i.e. $H^{db} \leqslant H^b$.

In the special case of an associated flow rule it follows rather trivially from eqn (33) that uniqueness is guaranteed only in the hardening regime. In this case Hill (1958, 1979), using the concept of a linear comparison solid, also showed that elastic/plastic bifurcation can never precede plastic/plastic bifurcation in analogy with the result above. Using virtually the same approach, Raniecki (1979) and Raniecki and Bruhns (1981) extended the results to general non-associated flow rules. Runesson and Mroz (1989), using a more direct approach based on the explicit spectral properties of the tangent stiffness tensor D_{ijkl} rather than using the (unnecessary) concept of a linear comparison solid, confirmed and extended the results in Raniecki and Bruhns (1981). For example, the explicit expression of H^b was re-established.

Limit points occur when D_{ijkl} becomes singular. Let H^l denote the corresponding value of the hardening modulus. It was shown in Runesson and Mroz (1989) that $H^l = 0$ holds independently of the amount of non-associativeness involved, although the corresponding eigendirections are affected. Furthermore, it was noted that $\mu^{(1)}$ (the smallest eigenvalue of D_{ijkl}) is always increasing with H. Since $\mu^{(1)} \geqslant \mu_s^{(1)}$ (where $\mu_s^{(1)}$ is the smallest eigenvalue of the symmetric part of D_{ijkl}), it follows immediately that $H^b \geqslant H^l = 0$.

In the special case of associated plasticity, where D_{ijkl} is symmetrical, it follows that the quadratic form in eqn (33) vanishes only when D_{ijkl} is singular, i.e. $H^b = H^l = 0$. We may then conclude that discontinuities bifurcations can never occur in the hardening regime, i.e. $H^{db} \leqslant 0$. This result is clearly generally valid with respect to the character of the elastic properties, whereas Rice (1976) proved this inequality only for isotropic elasticity.

We may now summarize the results pertinent to non-associated plasticity, see also Fig. 1a,

$$H^b \geqslant H^{db}, \quad H^b \geqslant H^l = 0. \tag{34}$$

While $H^b \geqslant 0$ always holds, nothing can be said in general about the sign of H^{db}.

In the case of associated plasticity we summarize:

$$0 = H^l = H^b \geqslant H^{db} \tag{35}$$

which is shown in Fig. 1b.

Remark: In Figs 1a, b we have also introduced the value H^{se} that corresponds to loss of strong ellipticity. Strong ellipticity is defined by $c_iQ_{ij}c_j > 0$ for arbitrary $c_i \neq 0$. From arguments that are similar to those related to D_{ijkl}, we conclude that $H^{se} \geqslant H^{db}$. On the other hand, when eqn (33) is satisfied strong ellipticity holds, whereas the converse is not true. In conclusion we have $H^b \geqslant H^{se} \geqslant H^{db}$.

6. CRITICAL BIFURCATION DIRECTIONS AND HARDENING MODULUS

Expressions for the critical bifurcation direction and hardening modulus have been obtained by Rudnicki and Rice (1975) for a non-associated Drucker–Prager material. We shall here present analytical solutions for a much broader class of materials, which includes the non-associated Drucker–Prager material as a special case. In particular, dependence on

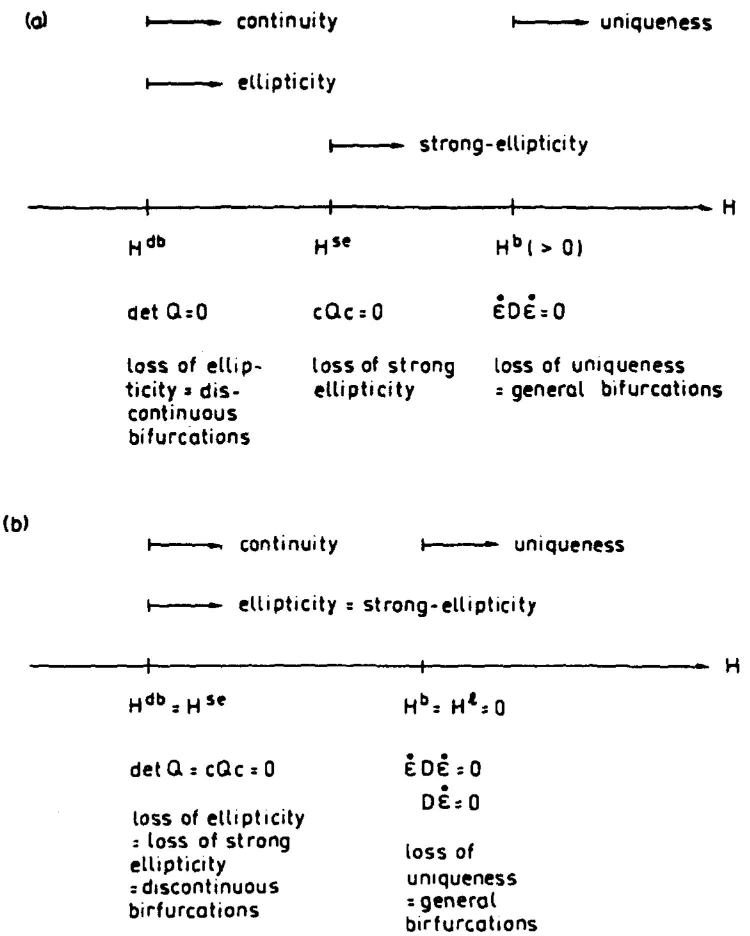

Fig. 1. Characteristics of the stress–strain response for (a) non-associated plasticity, (b) associated plasticity.

all three stress invariants is allowed, and mixed isotropic–kinematic hardening is considered. However, we shall assume that the elastic stiffness tensor D^e_{ijkl} is isotropic, i.e.

$$D^e_{ijkl} = 2G\left[\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right] \tag{36}$$

where G is the shear modulus and ν is Poisson's ratio. The elastic characteristic tensor Q^e_{il}, as defined by eqn (14), and its inverse P^e_{il} become

$$Q^e_{il} = G\left(\frac{1}{1-2\nu}n_i n_l + \delta_{il}\right); \quad P^e_{il} = \frac{1}{G}\left(-\frac{1}{2(1-\nu)}n_i n_l + \delta_{il}\right). \tag{37}$$

With these expressions H given by eqn (25) takes the form

$$\frac{H}{2G} = 2n_k f_{kl} g_{lj} n_j + \frac{\nu}{1-\nu} n_i (g_{ss} f_{ij} + f_{ss} g_{ij}) n_j - f_{ij} g_{ij} - \frac{\nu}{1-\nu} f_{ii} g_{ss} - \frac{1}{1-\nu} n_i f_{ij} n_j n_k g_{kl} n_l. \tag{38}$$

It turns out to be convenient to split f_{ij} and g_{ij} into their deviatoric and volumetric parts. For the volumetric part we introduce the notations f_v and g_v

$$f_v = f_{ii}, \quad g_v = g_{ii} \tag{39}$$

while the deviatoric parts of f_{ij} and g_{ij} are denoted by $\bar{f}_{ij}$ and $\bar{g}_{ij}$ respectively, i.e.

$$\bar{f}_{ij} = f_{ij} - \tfrac{1}{3}\delta_{ij} f_v, \quad \bar{g}_{ij} = g_{ij} - \tfrac{1}{3}\delta_{ij} g_v. \tag{40}$$

With this notation, eqn (38) can be written as

$$\frac{H}{4G} = n_i[\bar{f}_{il}\bar{g}_{lj} + \varphi f_v \bar{g}_{ij} + \varphi g_v \bar{f}_{ij}]n_j - \psi n_i \bar{f}_{ij} n_j n_k \bar{g}_{kl} n_l - \tfrac{1}{2}\bar{f}_{ij}\bar{g}_{ij} - \tfrac{2}{3}\varphi f_v g_v \tag{41}$$

where

$$\psi = \frac{1}{2(1-\nu)}, \quad \varphi = \frac{1+\nu}{6(1-\nu)}. \tag{42}$$

In order to obtain analytical solutions, we shall only consider situations where f_{ij} and g_{ij} possess identical principal directions. For quite a general class of plasticity models we define the yield and potential functions as $F = F(\sigma_{ij}, \alpha_{ij}, \kappa_\alpha)$ and $G = G(\sigma_{ij}, \alpha_{ij}, \kappa_\alpha)$, where α_{ij} is a tensorial hardening function and κ_α ($\alpha = 1, 2, \ldots$) are scalar hardening parameters. For example, if $F = F(\sigma_{ij} - \alpha_{ij}, \kappa_\alpha)$ and $G = G(\sigma_{ij} - \alpha_{ij}, \kappa_\alpha)$ are chosen as isotropic functions of the back-stress tensor $\sigma_{ij} - \alpha_{ij}$, then f_{ij} and g_{ij} will have identical principal directions, which of course coincide with those of $\bar{f}_{ij}$ and $\bar{g}_{ij}$. We observe that such models include isotropic and kinematic hardening as well as mixed isotropic-kinematic hardening.

Choosing the coordinate system colinear with these principal directions, we have

$$\bar{f}_{ij} = \begin{bmatrix} \bar{f}_1 & 0 & 0 \\ 0 & \bar{f}_2 & 0 \\ 0 & 0 & \bar{f}_3 \end{bmatrix}; \quad \bar{g}_{ij} = \begin{bmatrix} \bar{g}_1 & 0 & 0 \\ 0 & \bar{g}_2 & 0 \\ 0 & 0 & \bar{g}_3 \end{bmatrix} \tag{43}$$

where $\bar{f}_1, \bar{f}_2, \bar{f}_3$ and $\bar{g}_1, \bar{g}_2, \bar{g}_3$ denote the corresponding principal values, and the labelling of the axes is chosen so that $\bar{f}_1 \geqslant \bar{f}_2 \geqslant \bar{f}_3$. Furthermore, we shall assume that the volumetric strains will generally obey a non-associated flow rule, whereas the deviatoric strains are associated, i.e. while $\bar{f}_i = \bar{g}_i$ we allow that $f_v \neq g_v$.

With these assumptions, eqn (41) becomes

$$\frac{H}{4G} = (\bar{f}_1^2 + r\bar{f}_1)n_1^2 + (\bar{f}_2^2 + r\bar{f}_2)n_2^2 + (\bar{f}_3^2 + r\bar{f}_3)n_3^2 - \psi(\bar{f}_1 n_1^2 + \bar{f}_2 n_2^2 + \bar{f}_3 n_3^2)^2 - k \tag{44}$$

where

$$r = \varphi(f_v + g_v), \quad k = \tfrac{1}{2}(\bar{f}_1^2 + \bar{f}_2^2 + \bar{f}_3^2) + \tfrac{2}{3}\varphi f_v g_v. \tag{45}$$

Using Lagrange's multiplier method we can evaluate the extremum properties of

$$L = \frac{H}{4G} - \lambda(n_1^2 + n_2^2 + n_3^2 - 1) \tag{46}$$

where λ is a Lagrangian multiplier. Details of the analysis are given in the Appendix. We shall summarize the results by considering separately two cases: $r \geqslant 0$ and $r \leqslant 0$.

With tension defined positive, the case $r \geqslant 0$ is pertinent to a frictional material ($f_v > 0$) showing dilatant behavior ($g_v \geqslant 0$), whereas the case $r \leqslant 0$ may represent a capmodel ($f_v \leqslant 0$) corresponding to contractant behavior ($g_v \leqslant 0$). The relevant situations are shown

Table 1. Discontinuous bifurcation solutions for $r \geqslant 0$

$r \geqslant 0$	$c_{31} = \bar{f}_3 + (1-2\psi)\bar{f}_1 + r \leqslant 0$	$c_{31} = \bar{f}_3 + (1-2\psi)\bar{f}_1 + r \geqslant 0$
$\bar{f}_1 > \bar{f}_2 > \bar{f}_3$	$n_1^2 = c_{13}/2\psi(\bar{f}_1 - \bar{f}_3)$, $n_2 = 0$, $n_3^2 = -c_{31}/2\psi(\bar{f}_1 - \bar{f}_3)$ $H^{db}/4G = 1/4\psi(\bar{f}_1 + \bar{f}_3 + r)^2 - \bar{f}_1\bar{f}_3 - k$	$n_1^2 = 1$, $n_2 = n_3 = 0$ $H^{db}/4G = (1-\psi)\bar{f}_1^2 + r\bar{f}_1 - k$
$\bar{f}_1 = \bar{f}_2 > \bar{f}_3$	$n_1^2 + n_2^2 = c_{13}/2\psi(\bar{f}_1 - \bar{f}_3)$, $n_3^2 = -c_{31}/2\psi(\bar{f}_1 - \bar{f}_3)$ $H^{db}/4G$ as above	$n_1^2 + n_2^2 = 1$, $n_3 = 0$ $H^{db}/4G$ as above
$\bar{f}_1 > \bar{f}_2 = \bar{f}_3$	$n_1^2 = c_{13}/2\psi(\bar{f}_1 - \bar{f}_3)$, $n_2^2 + n_3^2 = -c_{31}/2\psi(\bar{f}_1 - \bar{f}_3)$ $H^{db}/4G$ as above	$n_1^2 = 1$, $n_2 = n_3 = 0$ $H^{db}/4G$ as above
$\bar{f}_1 = \bar{f}_2 = \bar{f}_3 = 0$	$n_1^2 + n_2^2 + n_3^2 = 1$ $H^{db}/4G = -k$	$n_1^2 + n_2^2 + n_3^2 = 1$ $H^{db}/4G = -k$

in Table 1 and Fig. 2 for $r \geqslant 0$, and in Table 2 and Fig. 2 for $r \leqslant 0$, where it has been assumed that $0 \leqslant \nu < 0.5$.

It appears that principally different situations are obtained depending on the sign of the quantities c_{31} and c_{13} defined by

$$c_{31} = \bar{f}_3 + (1-2\psi)\bar{f}_1 + r, \quad c_{13} = \bar{f}_1 + (1-2\psi)\bar{f}_3 + r. \tag{47}$$

Consider first the case where $r \geqslant 0$. The condition $c_{31} \leqslant 0$ is equivalent to

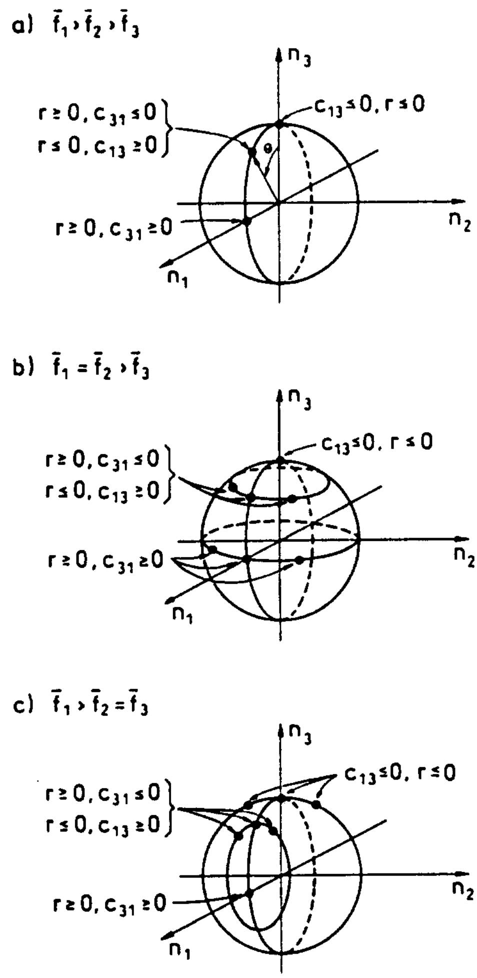

Fig. 2. Graphical representation of bifurcation directions. The corresponding symmetric solutions are not shown.

Table 2. Discontinuous bifurcation solutions for $r \leqslant 0$

$r \leqslant 0$	$c_{13} = \bar{f}_1 + (1-2\psi)\bar{f}_3 + r \geqslant 0$	$c_{13} = \bar{f}_1 + (1-2\psi)\bar{f}_3 + r \leqslant 0$
$\bar{f}_1 > \bar{f}_2 > \bar{f}_3$	$n_1^2 = c_{13}/2\psi(\bar{f}_1 - \bar{f}_3)$, $n_2 = 0$, $n_3^2 = c_{31}/2\psi(\bar{f}_1 - \bar{f}_3)$ $H^{db}/4G = 1/4\psi(\bar{f}_1 + \bar{f}_3 + r)^2 - \bar{f}_1\bar{f}_3 - k$	$n_1 = n_2 = 0$, $n_3^2 = 1$ $H^{db}/4G = (1-\psi)\bar{f}_3^2 + r\bar{f}_3 - k$
$\bar{f}_1 = \bar{f}_2 > \bar{f}_3$	$n_1^2 + n_2^2 = c_{13}/2\psi(\bar{f}_1 - \bar{f}_3)$, $n_3^2 = -c_{31}/2\psi(\bar{f}_1 - \bar{f}_3)$ $H^{db}/4G$ as above	$n_1 = n_2 = 0$, $n_3^2 = 1$ $H^{db}/4G$ as above
$\bar{f}_1 > \bar{f}_2 = \bar{f}_3$	$n_1^2 = c_{13}/2\psi(\bar{f}_1 - \bar{f}_3)$, $n_2^2 + n_3^2 = -c_{31}/2\psi(\bar{f}_1 - \bar{f}_3)$ $H^{db}/4G$ as above	$n_1 = 0, n_2^2 + n_3^2 = 1$ $H^{db}/4G$ as above
$\bar{f}_1 = \bar{f}_2 = \bar{f}_3 = 0$	$n_1^2 + n_2^2 + n_3^2 = 1$ $H^{db}/4G = -k$	$n_1^2 + n_2^2 + n_3^2 = 1$ $H^{db}/4G = -k$

$$2\bar{f}_1 + (2-3v)\bar{f}_2 - \bar{f}_3 \geqslant \frac{1+v}{2}(f_v + g_v) \tag{48}$$

and the corresponding maximum value H^{db} is given as

$$\frac{H^{db}}{4G} = \frac{1+v}{2}\left[\frac{(f_v - g_v)^2}{18(1-v)} - \left(\bar{f}_2 + \frac{f_v + g_v}{6}\right)^2\right]. \tag{49}$$

The bifurcation directions for the case $\bar{f}_1 > \bar{f}_2 > \bar{f}_3$ are defined by

$$\tan^2\theta = \frac{n_1^2}{n_3^2} \tag{50}$$

where θ denotes the angle in the x_1, x_3-plane from the x_3-axis to the normal vector $(n_1, 0, n_3)$, cf. Fig. 2. Using the expressions given in Table 1 for n_1^2 and n_3^2 and defining, in accordance with Rudnicki and Rice (1975), the parameter ξ as

$$\xi = \frac{1+v}{3}(f_v + g_v) - 2(1-v)\bar{f}_2 \tag{51}$$

we obtain

$$\tan^2\theta = \frac{\xi - 2\bar{f}_3}{2\bar{f}_1 - \xi}. \tag{52}$$

The other situation defined by $C_{31} \geqslant 0$ (see Table 1) corresponds to the solution $\theta = 90°$ and

$$\frac{H^{db}}{4G} = -\frac{1+v}{1-v}\left[\frac{(\bar{f}_2 - \bar{f}_3)^2}{2(1+v)} + \bar{f}_2\bar{f}_3 + \frac{f_v + g_v}{6}(\bar{f}_2 + \bar{f}_3) + \tfrac{1}{9}f_v g_v\right]. \tag{53}$$

Consider next the case where $r \leqslant 0$. The relevant situations are shown in Table 2 and Fig. 2 and it appears that the principal situations are obtained depending on the sign of c_{13}. Equivalently, to eqn (48), the condition $c_{13} \geqslant 0$ can be reformulated

$$2\bar{f}_3+(2-3\nu)\bar{f}_2-\bar{f}_1 \leqslant \frac{1+\nu}{2}(f_v+g_v). \tag{54}$$

For $\bar{f}_1>\bar{f}_2>\bar{f}_3$, the corresponding value H^{db} and the angle θ are again given by eqns (49) and (50) respectively.

The other situation defined by $c_{13} \leqslant 0$ (see Table 1) corresponds to the solution $\theta = 0^\circ$ and

$$\frac{H^{db}}{4G} = -\frac{1+\nu}{1-\nu}\left[\frac{(\bar{f}_1-\bar{f}_2)^2}{2(1+\nu)}+\bar{f}_1\bar{f}_2+\frac{f_v+g_v}{6}(\bar{f}_1+\bar{f}_2)+\tfrac{1}{9}f_vg_v\right]. \tag{55}$$

Finally, we note that in the case $\bar{f}_1=\bar{f}_2=\bar{f}_3=0$, which is of interest for a degenerated cap model, the bifurcation directions are arbitrary with

$$\frac{H^{db}}{4G} = -\frac{1+\nu}{9(1-\nu)}f_vg_v. \tag{56}$$

7. EXAMPLES OF MATERIAL BEHAVIOR

Drucker–Prager criterion

One of the simplest models for describing the behavior of frictional materials is defined by

$$F = \sqrt{J_2}+\frac{\mu}{3}I_1-c = 0, \quad G = \sqrt{J_2}+\frac{\beta}{3}I_1-c_g = 0 \tag{57}$$

where μ and β are parameters that represent the angle of internal friction and dilatancy respectively. This non-associated Drucker–Prager material model was treated by Rudnicki and Rice (1975). The invariants J_2 and I_1 are given by $J_2 = s_{ij}s_{ij}/2$ and $I_1 = \sigma_{ii}$ respectively, where s_{ij} denotes the deviatoric stress tensor. This gives

$$f_v = \mu, g_v = \beta \quad \text{and} \quad \bar{f}_i = \frac{N_i}{2} \tag{58}$$

where $N_i = s_i/\sqrt{J_2}$ and s_i are the deviatoric principal stresses: $s_1 \geqslant s_2 \geqslant s_3$. With these expressions, the solutions given by eqns (49) and (52) reduce to the results obtained by Rudnicki and Rice (1975). The case considered in Rudnicki and Rice (1975) was defined by $\mu \geqslant 0$, $\beta \geqslant 0$, i.e. $r \geqslant 0$. The condition for the validity of the solutions (49) and (52) is then given by eqn (48), which in the present case can be expressed as

$$2N_1+(2-3\nu)N_2-N_3 \geqslant (1+\nu)(\mu+\beta). \tag{59}$$

We note that the equivalent form of this condition given in Rudnicki and Rice (1975) is

$$2N_1-N_2-N_3 \geqslant 2(\mu+\beta) \tag{60}$$

which is obtained from the condition that the critical bifurcation direction is characterized by the solution $n_2 = 0$ rather than $n_3 = 0$. However, our condition (59) arises from the requirement that the solution satisfies $0 \leqslant n_1^2 \leqslant 1$, which follows from Table 1, and the fact that $c_{31} \leqslant 0$ implies that $c_{13} \leqslant 2\psi(\bar{f}_1-\bar{f}_3)$. The condition (60) is different from eqn (59) except when $\nu = 1$, which is not physically valid. This means that the results given by Rudnicki and Rice (1975), in their Table 1, are confirmed except for certain categories of the axially-symmetric compression data ($N_1 = N_2 = 1/\sqrt{3}$, $N_3 = -2/\sqrt{3}$). In this case, when $\nu = 0.3$ the condition (59) is satisfied for $\mu+\beta \leqslant 2.26$, while condition (60) is satisfied only for $\mu+\beta \leqslant \sqrt{3}/2 \simeq 0.87$. This explains why the solution (49) is rejected for $\mu+\beta \geqslant 0.87$

in Table 1 of Rudnicki and Rice (1975). However, according to our condition (59) the solution given by (49) is acceptable in the total range of $\mu+\beta$ considered in Table 1 of Rudnicki and Rice (1975).

Von Mises criterion

An associated isotropic von Mises model is obtained from eqn (57) by setting $\mu = \beta = 0$, which gives $r = 0$. The condition (59) then reads

$$-\nu N_1+(1+\nu)N_3 \leqslant 0 \tag{61}$$

which is always satisfied as $N_1 > 0$ and $N_3 < 0$. Therefore, only the solution given by eqns (49) and (52) is relevant and the extreme cases $\theta = 0$ and $\theta = 90°$ can never occur. The solution is obtained as

$$\tan^2\theta = \frac{(1-\nu)N_1-\nu N_3}{\nu N_1-(1-\nu)N_3}, \quad \frac{H^{db}}{4G} = -\frac{1+\nu}{8}N_2^2. \tag{62}$$

For uniaxial tension ($\sigma_1 > 0$, $\sigma_2 = \sigma_3 = 0$) we obtain

$$\tan^2\theta = \frac{2-\nu}{1+\nu}, \quad \frac{H^{db}}{4G} = -\frac{1+\nu}{24} \tag{63}$$

which for $\nu = 0$ results in $\theta = 54.7°$ and for $\nu = 0.3$ yields $\theta = 48.8°$. However, it is of interest that such bifurcations require that H^{db} assumes the large negative value $H^{db} = -E/12$ according to eqn (63).

Rankine criterion

The maximum tension cut-off criterion with an associated flow rule is defined by

$$F = G = \sigma_1-\sigma_t = 0 \tag{64}$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ (tension positive) and $\sigma_t > 0$ is the uniaxial tensile yield stress. Equation (64) implies that

$$f_v = g_v = 1 \quad \text{and} \quad \bar{f}_1 = \tfrac{2}{3},\ \bar{f}_2 = \bar{f}_3 = -\tfrac{1}{3}. \tag{65}$$

It follows that $r > 0$ and $c_{31} = 0$ always hold. Therefore, both the H^{db} values given by eqns (49) and (53) are applicable and both results in $H^{db} = 0$, i.e. bifurcation is possible as soon as the perfectly plastic state is reached. Furthermore, from Table 1 follows that both the expressions for the slip directions coincide: $n_1^2 = 1$, $n_2 = n_3 = 0$, i.e. $\theta = 90°$. These properties of the maximum tension cut-off criterion suggest its use for crack modelling in cementitious materials and the related bifurcation properties are investigated in Ottosen (1986) and Barré and Maier (1988).

Coulomb criterion

A Coulomb material with an associated flow rule is defined by

$$F = G = \tfrac{1}{2}(\sigma_1-\sigma_3)+\tfrac{1}{2}(\sigma_1+\sigma_3)\sin\Phi-c\cos\Phi = 0 \tag{66}$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ (tension positive), Φ is the friction angle and c is the cohesion. This criterion implies that

$$f_v = g_v = \sin\Phi \tag{67}$$

and

$$\bar{f}_1 = \tfrac{1}{2}(1+\tfrac{1}{3}\sin\Phi),\quad \bar{f}_2 = -\tfrac{1}{3}\sin\Phi,\quad \bar{f}_3 = \tfrac{1}{2}(-1+\tfrac{1}{3}\sin\Phi) \tag{68}$$

where the order $\bar{f}_1 > \bar{f}_2 > \bar{f}_3$ holds because of the relation $-1 < \sin\Phi < 1$. Tresca's criterion is obtained when $\Phi = 0$. For $\sin\Phi = 1$, we have $\bar{f}_1 > \bar{f}_2 = \bar{f}_3$ and the criterion reduces to Rankine's criterion treated previously. We shall therefore in the following assume that $-1 < \sin\Phi < 1$. It appears that $r = 2\varphi\sin\Phi$ and that c_{31} and c_{13} given by eqn (47) become

$$c_{31} = -\frac{1-\sin\Phi}{2(1-\nu)},\quad c_{13} = \frac{1+\sin\Phi}{2(1-\nu)}. \tag{69}$$

We recall that $\bar{f}_1 > \bar{f}_2 > \bar{f}_3$ always holds. Therefore, both when $0 \leqslant \sin\Phi < 1$ (implying $r \geqslant 0$ and $c_{31} < 0$) and when $-1 < \sin\Phi \leqslant 0$ (implying $r \leqslant 0$ and $c_{13} > 0$), the only situation of interest is given by eqns (49) and (52), which give

$$\tan^2\theta = \frac{1+\sin\Phi}{1-\sin\Phi} = \tan^2\left(45^\circ + \frac{\Phi}{2}\right),\quad H^{dh} = 0. \tag{70}$$

This result confirms the classical hypothesis of Mohr that when the perfectly plastic state has been reached, the slip planes (bifurcation planes) have the orientation $\theta = \pm(45^\circ + \Phi/2)$. It may be noticed that our only assumptions are those of associated plasticity and the Coulomb criterion, while the classical approach by Mohr also involves the *postulate* that slip planes develop so that $\theta = (45^\circ + \Phi/2)$ at the state of failure (when $H^{dh} = 0$). In our analysis, the latter result is rather obtained from the bifurcation analysis.

8. BIFURCATED STRAIN RATE FIELD

The bifurcated strain rate $[\dot{\varepsilon}_{ij}]$ given by eqn (7) is in general characterized as a plane strain: Since one of the rows can always be determined as a linear combination of the remaining two rows, this indicates that at least one eigenvalue of $[\dot{\varepsilon}_{ij}]$ is zero. The character of $[\dot{\varepsilon}_{ij}]$ may be evaluated in more detail, when the critical directions n_i are known. In the present case of isotropic elasticity and non-associated volumetric flow it follows from eqns (26), (37) and (36) that the pertinent eigenvector c_i is given as

$$c_i = 2\gamma[\bar{g}_{ij}n_j + \varphi g_v n_i - \psi n_i \bar{g}_{ij} n_j n_i]. \tag{71}$$

Expressed in terms of principal directions of $\bar{g}_{ij}$ eqn (71) can be rewritten as (with the current assumption that $\bar{g}_{ij} = \bar{f}_{ij}$)

$$c_i = 2\gamma(\bar{f}_i + \varphi g_v - \psi m)n_i,\quad i = 1,2,3 \text{ (no summation)} \tag{72}$$

where

$$m = \bar{f}_1 n_1^2 + \bar{f}_2 n_2^2 + \bar{f}_3 n_3^2. \tag{73}$$

We shall now evaluate the strain rate field for the particular case that $r \geqslant 0$, i.e. Table 1 applies.

For the case that $\bar{f}_1 > \bar{f}_2 > \bar{f}_3$ and $c_{31} \leqslant 0$, i.e. $n_1 \neq 0$, $n_2 = 0$, $n_3 \neq 0$, eqn (72) yields $c_2 = 0$ implying that the bifurcated strain rate field has the following plane strain property:

$$[\dot{\varepsilon}_{ij}] = \frac{1}{2}\begin{bmatrix} 2c_1n_1 & 0 & c_1n_3 + c_3n_1 \\ 0 & 0 & 0 \\ c_1n_3 + c_3n_1 & 0 & 2c_3n_3 \end{bmatrix}. \tag{74}$$

The condition $n_2 = 0$ means that the principal axis x_2 is located in the bifurcation plane, and from eqn (74) we conclude that the normal strain rate $[\dot{\varepsilon}_{22}]$, as well as the shear components $[\dot{\varepsilon}_{21}]$ and $[\dot{\varepsilon}_{23}]$, vanish.

It may be of interest to evaluate eqn (74) for the Coulomb material given by eqn (66). In this case we obtain from Table 1

$$n_1^2 = \tfrac{1}{2}(1+\sin\Phi), \quad n_2 = 0, \quad n_3^2 = \tfrac{1}{2}(1-\sin\Phi) \tag{75}$$

and from eqn (72)

$$c_1 = \gamma n_1, \quad c_2 = 0, \quad c_3 = -\gamma n_3 \tag{76}$$

which inserted into eqn (74) gives

$$[\dot{\varepsilon}_{ij}] = \frac{\gamma}{2}\begin{bmatrix} 1+\sin\Phi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(1-\sin'\Phi) \end{bmatrix}. \tag{77}$$

For the Tresca material, which is defined by $\Phi = 0$, this reduces to a state of pure shear.

In the case that $\bar{J}_1 > \bar{J}_2 > \bar{J}_3$ and $c_{31} \geqslant 0$, i.e. $n_1^2 = 1$, $n_2 = n_3 = 0$, eqn (72) yields $c_2 = c_3 = 0$, and so eqn (74) yields

$$[\dot{\varepsilon}_{ij}] = \begin{bmatrix} \pm c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{78}$$

which is a uniaxial state of strain. As $n_2 = n_3 = 0$, the bifurcation plane is defined by the plane spanned by $\bar{J}_2$ and $\bar{J}_3$, and the normal strain rate discontinuity $[\dot{\varepsilon}_{11}] = \pm c_1$ is directed normal to the bifurcation plane. This uniaxial state is, therefore, occasionally termed a splitting mode and follows in particular, when the maximum tension cut-off criterion is adopted.

9. CONCLUSIONS

From a spectral analysis of the characteristic tangent stiffness modulus tensor, the discontinuous bifurcation properties were established explicitly. It was shown that the classical expression for the critical hardening modulus given by Rice (1976) is, in fact, the only possible one. The spectral analysis provides a straightforward assessment of issues regarding elastic/plastic versus plastic/plastic localization in a fashion that is similar to the treatment of the general bifurcation problem as discussed by Runesson and Mroz (1989). Furthermore, it turns out that the spectral properties presented here form the basis for an explicit evaluation of the related acceleration wave problem, cf. Ottosen and Runesson (1990).

The main result of the paper consists of the explicit analytical expressions for the critical bifurcation direction and the corresponding hardening modulus, that were established for quite a broad class of elastic–plastic models. These analytical results contain the previous expressions by Rudnicki and Rice (1975) as a special case although with a slight difference for the range of validity of these results, where the range given in Rice and Rudnicki (1980) seems to be incorrect. Finally, the directions of the slip planes for an associated Coulomb material were shown to be in accordance with the classical postulate by Mohr.

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APPENDIX: DETERMINATION OF BIFURCATION SOLUTIONS

The necessary conditions for a stationary value of the Lagrangian L in eqn (46) are

$$\frac{\partial L}{\partial n_i} = 2A_i n_i = 0 \text{ (no summation)}, \quad \frac{\partial L}{\partial \lambda} = Y = -(n_1^2+n_2^2+n_3^2-1) = 0 \tag{A1}$$

where

$$A_i = \bar{f}_i^2 + r\bar{f}_i - 2\psi m\bar{f}_i - \lambda, \quad m = \bar{f}_1 n_1^2 + \bar{f}_2 n_2^2 + \bar{f}_3 n_3^2. \tag{A2}$$

The symmetric Hessian H_{ij} of L is

$$H_{ij} = \frac{\partial^2 L}{\partial n_i \partial n_j} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \tag{A3}$$

where (no summation)

$$H_{ii} = 2A_i - 8\psi(\bar{f}_i n_i)^2, \quad H_{ij} = -8\psi \bar{f}_i \bar{f}_j n_i n_j \; (i \neq j). \tag{A4}$$

To establish necessary conditions for the existence of a maximum we introduce the concept of a tangent plane. The constraint condition is given by $Y = 0$, cf. eqn (A1), and the tangent plane is then defined by the vectors n_i^* satisfying $(\partial Y/\partial n_i)n_i^* = 0$, i.e.

$$n_i n_i^* = 0. \tag{A5}$$

In this expression n_i is the vector that satisfies the conditions for an extremum and the vectors n_i^* are then defined by eqn (A5). According to Luenberger (1984, p. 306) the necessary conditions for the existence of a maximum are that eqn (A1) is fulfilled and that H_{ij} satisfies the inequality

$$n_i^* H_{ij} n_j^* \leq 0 \tag{A6}$$

for each n_i^* that satisfies eqn (A5).

With these preliminary remarks, we may now identify three cases in regard to eqn (A1): (i) none of n_1, n_2, n_3 is zero, (ii) one of n_1, n_2, n_3 is zero, and (iii) two of n_1, n_2, n_3 are zero.

Case (i). *None of* n_1, n_2, n_3 *is zero*

From eqn (A1) we conclude that $A_1 = A_2 = A_3 = 0$, which condition can be written as

$$m\bar{f}_i = \frac{1}{2\psi}(\bar{f}_i^2 + r\bar{f}_i - \lambda), \quad i = 1, 2, 3 \tag{A7}$$

where m is defined by eqn (A2). Equation (A7) can be written explicitly as

$$\begin{bmatrix} \bar{f}_1^2 & \bar{f}_1\bar{f}_2 & \bar{f}_1\bar{f}_3 \\ \bar{f}_1\bar{f}_2 & \bar{f}_2^2 & \bar{f}_2\bar{f}_3 \\ \bar{f}_1\bar{f}_3 & \bar{f}_2\bar{f}_3 & \bar{f}_3^2 \end{bmatrix} \begin{bmatrix} n_1^2 \\ n_2^2 \\ n_3^2 \end{bmatrix} = \frac{1}{2\psi} \begin{bmatrix} \bar{f}_1^2 + r\bar{f}_1 - \lambda \\ \bar{f}_2^2 + r\bar{f}_2 - \lambda \\ \bar{f}_3^2 + r\bar{f}_3 - \lambda \end{bmatrix} \tag{A8}$$

As $A_1 = A_2 = A_3 = 0$, the Hessian H_{ij} takes the form

$$H_{ij} = -8\psi \begin{bmatrix} \bar{f}_1^2 n_1^2 & \bar{f}_1\bar{f}_2 n_1 n_2 & \bar{f}_1\bar{f}_3 n_1 n_3 \\ \bar{f}_1\bar{f}_2 n_1 n_2 & \bar{f}_2^2 n_2^2 & \bar{f}_2\bar{f}_3 n_2 n_3 \\ \bar{f}_1\bar{f}_3 n_1 n_3 & \bar{f}_2\bar{f}_3 n_2 n_3 & \bar{f}_3^2 n_3^2 \end{bmatrix} = -8\psi h_i h_j \tag{A9}$$

where $h_i = \bar{f}_i n_i$ (no summation). Since H_{ij} is negative semi-definite in this particular case, it appears that inequality (A6) is always satisfied, implying that all solutions of eqn (A1) are also possible maximum points.

Case (i1). Assume $\bar{f}_1 = 0$. This requires $\bar{f}_2 = -\bar{f}_3$ and, as $0 \geq \bar{f}_2 \geq \bar{f}_3$, we conclude that $\bar{f}_1 = \bar{f}_2 = \bar{f}_3 = 0$. It then follows from eqn (A7) that $\lambda = 0$ and that any vector satisfying

$$n_1^2 + n_2^2 + n_3^2 = 1, \tag{A10}$$

corresponds to a possible maximum.

Case (i2). Assume $\bar{f}_1 \geq \bar{f}_2 \geq \bar{f}_3$ and $\bar{f}_1 \neq 0$. From eqn (A7) we can determine the parameter m ($i = 1$) and use this expression in the remaining two relations to obtain

$$\lambda(\bar{f}_1 - \bar{f}_2) = -\bar{f}_1\bar{f}_2(\bar{f}_1 - \bar{f}_2), \quad \lambda(\bar{f}_1 - \bar{f}_3) = -\bar{f}_1\bar{f}_3(\bar{f}_1 - \bar{f}_3). \tag{A11}$$

Case (i2a). Assume $\bar{f}_1 > \bar{f}_2 > \bar{f}_3$. The two equations in (A11) cannot be satisfied simultaneously, implying that no maximum point exists in this case.

Case (i2b). Assume $\bar{f}_1 = \bar{f}_2 > \bar{f}_3$. It appears that the first equation in (A11) is satisfied identically, whereas the second equation provides

$$\lambda = -\bar{f}_1\bar{f}_3. \tag{A12}$$

As both $\bar{f}_1 \neq 0$ and $\bar{f}_3 \neq 0$, eqn (A12) implies that the only independent equation in (A8) is

$$\bar{f}_1(n_1^2 + n_2^2) + \bar{f}_3 n_3^2 = \frac{1}{2\psi}(\bar{f}_1 + \bar{f}_3 + r). \tag{A13}$$

Combining eqn (A13) with the constraint $n_1^2 + n_2^2 = 1 - n_3^2$, we then obtain

$$n_3^2 = -\frac{\bar{f}_3 + (1 - 2\psi)\bar{f}_1 + r}{2\psi(\bar{f}_1 - \bar{f}_3)}, \quad n_1^2 + n_2^2 = 1 - n_3^2. \tag{A14}$$

This solution is valid when $n_3^2 \geq 0$ and $n_3^2 \leq 1$, or

$$\bar{f}_3 + (1 - 2\psi)\bar{f}_1 + r \leq 0, \quad \bar{f}_1 + (1 - 2\psi)\bar{f}_3 + r \geq 0. \tag{A15}$$

Case (i2c). Assume $\bar{f}_1 > \bar{f}_2 = \bar{f}_3$. Both equations in (A11) give

$$\lambda = -\bar{f}_1\bar{f}_3. \tag{A16}$$

Similarly to Case (i2b), we obtain

$$n_1^2 = \frac{\bar{f}_1 + (1 - 2\psi)\bar{f}_3 + r}{2\psi(\bar{f}_1 - \bar{f}_3)}, \quad n_2^2 + n_3^2 = 1 - n_1^2. \tag{A17}$$

This solution requires that $n_1^2 \geq 0$ and $n_1^2 \leq 1$, or

$$\bar{f}_1+(1-2\psi)\bar{f}_3+r \geq 0, \quad \bar{f}_3+(1-2\psi)\bar{f}_1+r \leq 0. \tag{A18}$$

Case (ii). *One of* n_1, n_2, n_3 *is zero*

Case (iiA). Assume $n_1 = 0$ ($n_2 \neq 0$, $n_3 \neq 0$). From eqn (A1) we conclude that $A_2 = A_3 = 0$, and eqn (A2) gives

$$m\bar{f}_\alpha = \frac{1}{2\psi}(\bar{f}_\alpha^2+r\bar{f}_\alpha-\lambda), \quad \alpha = 2, 3. \tag{A19}$$

Since $A_2 = A_3 = 0$ the Hessian H_{ij} becomes

$$H_{ij} = \begin{bmatrix} 2A_1 & 0 & 0 \\ 0 & -8\psi\bar{f}_2^2n_2^2 & -8\psi\bar{f}_2\bar{f}_3n_2n_3 \\ 0 & -8\psi\bar{f}_2\bar{f}_3n_2n_3 & -8\psi\bar{f}_3^2n_3^2 \end{bmatrix} \tag{A20}$$

where eqn (A2) gives

$$A_1 = \bar{f}_1^2+r\bar{f}_1-2\psi m\bar{f}_1-\lambda. \tag{A21}$$

From the requirement (A6) we obtain with eqn (A20) the condition

$$2A_1n_1^{*2}-8\psi(\bar{f}_2n_2n_2^*+\bar{f}_3n_3n_3^*)^2 \leq 0. \tag{A22}$$

The vectors n_i^*, which determine the tangent plane, are given by eqn (A5). Since $n_1 = 0$, we obtain

$$n_2n_2^*+n_3n_3^* = 0, \quad n_1^* \text{ arbitrary}. \tag{A23}$$

A possible solution of eqn (A23) is $n_i^* = (n_1^*, 0, 0)$, where n_1^* is arbitrary. It then appears from eqn (A22) that the necessary condition for a maximum becomes

$$A_1 \leq 0. \tag{A24}$$

Case (iiA1). Assume $\bar{f}_1 \geq \bar{f}_2 > \bar{f}_3$ and $\bar{f}_3 < 0$. Eliminating m and solving for λ from eqn (A19), we obtain

$$\lambda = -\bar{f}_2\bar{f}_3, \tag{A25}$$

which inserted in eqn (A19) for $\alpha = 3$ gives

$$m = \bar{f}_2n_2^2+\bar{f}_3n_3^2 = \frac{1}{2\psi}(\bar{f}_2+\bar{f}_3+r). \tag{A26}$$

We may now evaluate the condition (A24) using eqns (A25) and (A26). This yields the condition

$$A_1 = (\bar{f}_1-\bar{f}_2)(\bar{f}_1-\bar{f}_3) \leq 0. \tag{A27}$$

Case (iiA1a). Assume $\bar{f}_1 > \bar{f}_2 > \bar{f}_3$. It appears immediately that the requirement (A27) is never satisfied in this case, i.e. no maximum point exists.

Case (iiA1b). Assume $\bar{f}_1 = \bar{f}_2 > \bar{f}_3$. The requirement (A27) is fulfilled and the constraint $n_2^2 = 1-n_3^2$ in combination with eqn (A26) results in the solution

$$n_3^2 = -\frac{\bar{f}_3+(1-2\psi)\bar{f}_1+r}{2\psi(\bar{f}_1-\bar{f}_3)}, \quad n_2^2 = 1-n_3^2. \tag{A28}$$

However, this solution can be viewed as a special case of solution (A14) when $n_1 = 0$.

Case (iiA2). Assume $\bar{f}_1 \geq \bar{f}_2 = \bar{f}_3$. Since $\bar{f}_2 = \bar{f}_3$, eqn (A2) gives $m = \bar{f}_3(n_2^2+n_3^2) = \bar{f}_3$ and it follows from eqn (A19) for $\alpha = 3$ that

$$\lambda = r\bar{f}_3+(1-2\psi)\bar{f}_3^2 \tag{A29}$$

and the solution is arbitrary within the constraint

$$n_2^2+n_3^2 = 1 \quad (n_1 = 0). \tag{A30}$$

To check whether this solution represents a possible maximum, we consider the criterion (A24) with eqn (A29) and $m = \bar{f}_3$ to obtain

$$A_1 = (\bar{f}_1 - \bar{f}_3)[\bar{f}_1 + (1-2\psi)\bar{f}_3 + r] \leqslant 0 \quad \text{(A31)}$$

which is always satisfied upon fulfillment of the condition

$$\bar{f}_1 + (1-2\psi)\bar{f}_3 + r \leqslant 0. \quad \text{(A32)}$$

In the special case when $\bar{f}_1 = \bar{f}_2 = \bar{f}_3 = 0$, eqn (A2) gives $A_1 = 0$ independently of whether the condition (A32) is satisfied or not, and the solution is defined in (A30).

Case (iiB). Assume $n_2 = 0$ ($n_1 \neq 0$, $n_3 \neq 0$). From eqn (A1) we conclude that $A_1 = A_3 = 0$ and eqn (A2) gives

$$m\bar{f}_\alpha = \frac{1}{2\psi}(\bar{f}_\alpha^2 + r\bar{f}_\alpha - \lambda), \quad \alpha = 1, 3. \quad \text{(A33)}$$

Since $A_1 = A_3 = 0$ the Hessian H_{ij} now becomes

$$H_{ij} = \begin{bmatrix} -8\psi\bar{f}_1^2 n_1^2 & 0 & -8\psi\bar{f}_1\bar{f}_3 n_1 n_3 \\ 0 & 2A_2 & 0 \\ -8\psi\bar{f}_1\bar{f}_3 n_1 n_3 & 0 & -8\psi\bar{f}_3^2 n_3^2 \end{bmatrix} \quad \text{(A34)}$$

where eqn (A2) gives

$$A_2 = \bar{f}_2^2 + r\bar{f}_2 - 2\psi m\bar{f}_2 - \lambda. \quad \text{(A35)}$$

In fashion similar to Case (iiA) we conclude that the condition for a maximum is

$$A_2 \leqslant 0. \quad \text{(A36)}$$

Case (iiB1). Assume $\bar{f}_1 \geqslant \bar{f}_2 \geqslant \bar{f}_3$ and $\bar{f}_3 < 0$, i.e. $\bar{f}_1 > \bar{f}_3$. Eliminating m and solving for λ from eqn (A33), we obtain

$$\lambda = -\bar{f}_1\bar{f}_3 \quad \text{(A37)}$$

which inserted in eqn (A33) for $\alpha = 3$ gives

$$m = \bar{f}_1 n_1^2 + \bar{f}_3 n_3^2 = \frac{1}{2\psi}(\bar{f}_1 + \bar{f}_3 + r). \quad \text{(A38)}$$

Inserting the constraint $n_1^2 = 1 - n_3^2$ into eqn (A38), we obtain the solution

$$n_3^2 = -\frac{\bar{f}_3 + (1-2\psi)\bar{f}_1 + r}{2\psi(\bar{f}_1 - \bar{f}_3)}, \quad n_1^2 = 1 - n_3^2. \quad \text{(A39)}$$

This solution is valid provided $n_3^2 \geqslant 0$ and $n_3^2 \leqslant 1$, i.e.

$$\bar{f}_3 + (1-2\psi)\bar{f}_1 + r \leqslant 0, \quad \bar{f}_1 + (1-2\psi)\bar{f}_3 + r \geqslant 0. \quad \text{(A40)}$$

We may now evaluate the condition (A36). Equations (A37) and (A38) inserted into eqn (A36) imply

$$A_2 = -(\bar{f}_1 - \bar{f}_2)(\bar{f}_2 - \bar{f}_3) \leqslant 0 \quad \text{(A41)}$$

which is always satisfied as we have assumed that $\bar{f}_1 \geqslant \bar{f}_2 \geqslant \bar{f}_3$. Consequently, the solution (A39) represents a possible maximum point.

Case (iiC). Assume $n_3 = 0$ ($n_1 \neq 0$, $n_2 \neq 0$). From eqn (A1) we conclude that $A_1 = A_2 = 0$ and eqn (A2) gives

$$m\bar{f}_\alpha = \frac{1}{2\psi}(\bar{f}_\alpha^2 + r\bar{f}_\alpha - \lambda), \quad \alpha = 1, 2. \quad \text{(A42)}$$

Since $A_1 = A_2 = 0$, the Hessian becomes

$$H_{ij} = \begin{bmatrix} -8\psi\bar{f}_1^2 n_1^2 & -8\psi\bar{f}_1\bar{f}_2 n_1 n_2 & 0 \\ -8\psi\bar{f}_1\bar{f}_2 n_1 n_2 & -8\psi\bar{f}_2^2 n_2^2 & 0 \\ 0 & 0 & 2A_3 \end{bmatrix} \quad \text{(A43)}$$

where eqn (A2) gives

$$A_3 = \bar{f}_3^2 + r\bar{f}_3 - 2\psi m\bar{f}_3 - \lambda. \quad \text{(A44)}$$

Similarly to the previous case, we may conclude that the condition for a possible maximum is

$$A_3 \leqslant 0. \tag{A45}$$

Case (iiC1). Assume $\bar{f}_1 > \bar{f}_2 \geqslant \bar{f}_3$. From eqn (A42) the parameter m is eliminated with the result

$$\lambda = -\bar{f}_1\bar{f}_2 \tag{A46}$$

which inserted into eqn (A42) for $\alpha = 1$ results in

$$m = \bar{f}_1 n_1^2 + \bar{f}_2 n_2^2 = \frac{1}{2\psi}(\bar{f}_1 + \bar{f}_2 + r). \tag{A47}$$

With eqn (A47), the condition (A45) gives

$$A_3 = (\bar{f}_1 - \bar{f}_3)(\bar{f}_2 - \bar{f}_3) \leqslant 0. \tag{A48}$$

Case (iiC1a). Assume $\bar{f}_1 > \bar{f}_2 > \bar{f}_3$. It appears that the condition (A48) can never be satisfied in this case.

Case (iiC1b). Assume $\bar{f}_1 > \bar{f}_2 = \bar{f}_3$. The requirement (A48) is fulfilled and with the constraint $n_2^2 = 1 - n_1^2$ inserted into (A47), we obtain the solution

$$n_1^2 = \frac{\bar{f}_1 + (1-2\psi)\bar{f}_3 + r}{2\psi(\bar{f}_1 - \bar{f}_3)}, \quad n_2^2 = 1 - n_1^2. \tag{A49}$$

However, this solution can be viewed as a special case of eqn (A17) when $n_3 = 0$.

Case (iiC2). Assume $\bar{f}_1 = \bar{f}_2 \geqslant \bar{f}_3$. Since $\bar{f}_1 = \bar{f}_2$, eqn (A2) gives $m = \bar{f}_1$, and from eqn (A42) we obtain

$$\lambda = r\bar{f}_1 + (1-2\psi)\bar{f}_1^2 \tag{A50}$$

and the solution is arbitrary within the constraint

$$n_1^2 + n_2^2 = 1 \quad (n_3 = 0). \tag{A51}$$

This solution represents a possible maximum when the condition (A45) is satisfied, i.e. when

$$\bar{f}_3 + (1-2\psi)\bar{f}_1 + r \geqslant 0. \tag{A52}$$

Case (iii). *Two of* n_1, n_2, n_3 *are zero*

Case (iiiA). Assume $n_1 = n_2 = 0, n_3^2 = 1$. From eqn (A1) we conclude that $A_3 = 0$. As $m = \bar{f}_3$, we then obtain from eqn (A2)

$$\lambda = r\bar{f}_3 + (1-2\psi)\bar{f}_3^2. \tag{A53}$$

Since $n_1 = n_2 = A_3 = 0$, the Hessian H_{ij} becomes

$$H_{ij} = \begin{bmatrix} 2A_1 & 0 & 0 \\ 0 & 2A_2 & 0 \\ 0 & 0 & -8\psi\bar{f}_3^2 \end{bmatrix}. \tag{A54}$$

The n_i^* vectors, which define the tangent plane, are given by eqn (A5). Since $n_1 = n_2 = 0$, we obtain

$$n_3^* = 0, \quad n_1^* \text{ and } n_2^* \text{ arbitrary}. \tag{A55}$$

The necessary condition for a maximum is then given from (A6).

$$A_1 n_1^{*2} + A_2 n_2^{*2} \leqslant 0 \tag{A56}$$

i.e. $A_1 \leqslant 0$ and $A_2 \leqslant 0$. With $m = \bar{f}_3$ and λ given by eqn (A53), we obtain the conditions

$$A_1 = (\bar{f}_1 - \bar{f}_3)[\bar{f}_1 + (1-2\psi)\bar{f}_3 + r] \leqslant 0, \quad A_2 = (\bar{f}_2 - \bar{f}_3)[\bar{f}_2 + (1-2\psi)\bar{f}_3 + r] \leqslant 0. \tag{A57}$$

Since $\bar{f}_1 \geqslant \bar{f}_2 \geqslant \bar{f}_3$ the condition $A_2 \leqslant 0$ is satisfied whenever $A_1 \leqslant 0$ is satisfied, and the necessary condition for a maximum becomes

$$\bar{f}_1 + (1-2\psi)\bar{f}_3 + r \leqslant 0. \tag{A58}$$

In the special case $\bar{f}_1 = \bar{f}_2 = \bar{f}_3 = 0$, the solution $n_1 = n_2 = 0, n_3^2 = 1$ always represents a possible maximum.

Case (iiiB). Assume $n_1 = n_3 = 0, n_2^2 = 1$. From eqn (A1) we conclude that $A_2 = 0$. As $m = \bar{f}_2$, we obtain from eqn (A2)

$$\lambda = r\bar{f}_2 + (1-2\psi)\bar{f}_2^2. \tag{A59}$$

Since $n_1 = n_3 = A_2 = 0$, the Hessian becomes

$$H_{ij} = \begin{bmatrix} 2A_1 & 0 & 0 \\ 0 & -8\psi\bar{f}_2^2 & 0 \\ 0 & 0 & 2A_3 \end{bmatrix}. \tag{A60}$$

Similarly to the previous case we conclude that $A_1 \leqslant 0$ and $A_3 \leqslant 0$. With $m = \bar{f}_2$ and λ given by eqn (A59) we obtain the conditions

$$A_1 = (\bar{f}_1 - \bar{f}_2)[\bar{f}_1 + (1-2\psi)\bar{f}_2 + r] \leqslant 0, \quad A_3 = (\bar{f}_3 - \bar{f}_2)[\bar{f}_3 + (1-2\psi)\bar{f}_2 + r] \leqslant 0. \tag{A61}$$

Case (iiiB1). Assume $\bar{f}_1 > \bar{f}_2 > \bar{f}_3$. From (A61) a possible maximum requires that

$$\bar{f}_1 + (1-2\psi)\bar{f}_2 + r \leqslant 0, \quad \bar{f}_3 + (1-2\psi)\bar{f}_2 + r \geqslant 0. \tag{A62}$$

Consequently, we must require that

$$-\bar{f}_3 - (1-2\psi)\bar{f}_2 \leqslant r \leqslant -\bar{f}_1 - (1-2\psi)\bar{f}_2 \tag{A63}$$

which implies that $\bar{f}_1 - \bar{f}_3 \leqslant 0$. However, this condition is never fulfilled for the case considered.

Case (iiiB2). Assume $\bar{f}_1 = \bar{f}_2 > \bar{f}_3$. The condition $A_3 \leqslant 0$ in (A61) becomes

$$\bar{f}_3 + (1-2\psi)\bar{f}_2 + r \geqslant 0. \tag{A64}$$

Case (iiiB3). Assume $\bar{f}_1 > \bar{f}_2 = \bar{f}_3$. The condition $A_1 \leqslant 0$ in (A61) becomes

$$\bar{f}_1 + (1-2\psi)\bar{f}_3 + r \leqslant 0. \tag{A65}$$

Case (iiiC). Assume $n_1^2 = 1, n_2 = n_3 = 0$. From eqn (A1) we conclude that $A_1 = 0$. As $m = \bar{f}_1$, we then obtain from eqn (A2)

$$\lambda = r\bar{f}_1 + (1-2\psi)\bar{f}_1^2. \tag{A66}$$

Since $n_2 = n_3 = A_1 = 0$, the Hessian H_{ij} becomes

$$H_{ij} = \begin{bmatrix} -8\psi\bar{f}_1^2 & 0 & 0 \\ 0 & 2A_2 & 0 \\ 0 & 0 & 2A_3 \end{bmatrix}. \tag{A67}$$

In analogy with previous arguments, it is required that $A_2 \leqslant 0$ and $A_3 \leqslant 0$. With $m = \bar{f}_1$ and λ given by eqn (A66), it follows that

$$A_2 = (\bar{f}_2 - \bar{f}_1)[\bar{f}_2 + (1-2\psi)\bar{f}_1 + r] \leqslant 0, \quad A_3 = (\bar{f}_3 - \bar{f}_1)[\bar{f}_3 + (1-2\psi)\bar{f}_1 + r] \leqslant 0. \tag{A68}$$

Since $\bar{f}_1 \geqslant \bar{f}_2 \geqslant \bar{f}_3$ the condition $A_2 \leqslant 0$ is satisfied whenever $A_3 \leqslant 0$ is satisfied, and the necessary condition for a maximum becomes

$$\bar{f}_3 + (1-2\psi)\bar{f}_1 + r \geqslant 0. \tag{A69}$$

Remark: It turns out that the choice between the different solutions frequently depends on the sign of the two parameters c_{31} and c_{13} defined by

$$c_{31} = \bar{f}_3 + (1-2\psi)\bar{f}_1 + r, \quad c_{13} = \bar{f}_1 + (1-2\psi)\bar{f}_3 + r. \tag{A70}$$

Since $1-2\psi = -\nu/(1-\nu)$ and since $0 \leqslant \nu \leqslant 0.5$ in practice, it appears that $-1 < 1-2\psi \leqslant 0$. Since $\bar{f}_1 \geqslant 0$ and $\bar{f}_3 \leqslant 0$ it then follows that

$$\bar{f}_3 + (1-2\psi)\bar{f}_1 \leqslant 0, \quad \bar{f}_1 + (1-2\psi)\bar{f}_3 \geqslant 0. \tag{A71}$$

Therefore, when $r \geqslant 0$, it appears that $c_{13} \geqslant 0$ (always), whereas c_{31} can take any value. Likewise, when $r \leqslant 0$, it appears that $c_{31} \leqslant 0$ (always), whereas c_{13} can take any value.